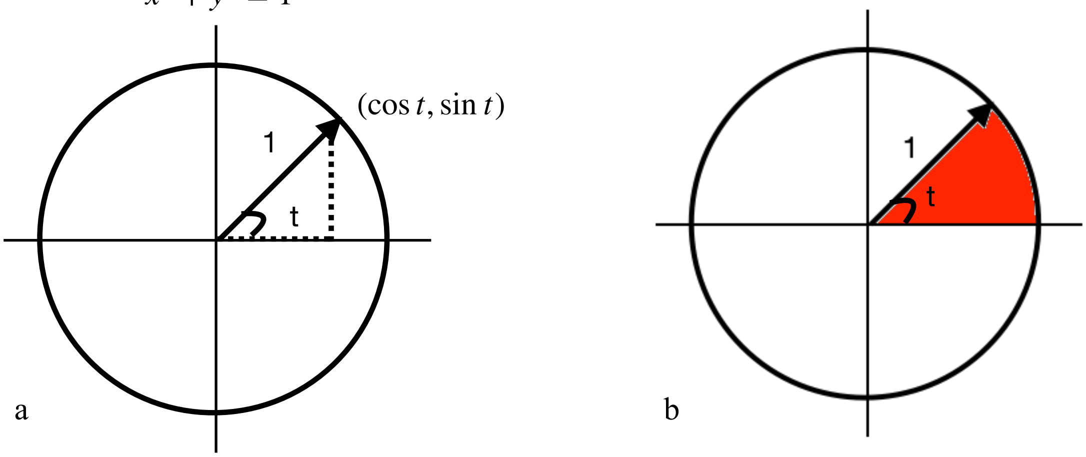
Hyperbolic Functions

Circular Trig Functions

$x^2 + y^2 = 1$

Figure 1



$\cos^2 t + \sin^2 t = 1$

Area of circle: πr^2

Portion of circumference included in red sector:

$$\frac{t}{2\pi r}$$

Area of sector in red:

$$\frac{t}{2\pi r} = \frac{x}{\pi r^2}$$

$$x = \frac{t \cdot \pi r^2}{2\pi r}$$

$$x = \frac{t \cdot r}{2} = \frac{t \cdot 1}{2}$$

$$x = \frac{t}{2}$$

Hyperbolic Trig Functions

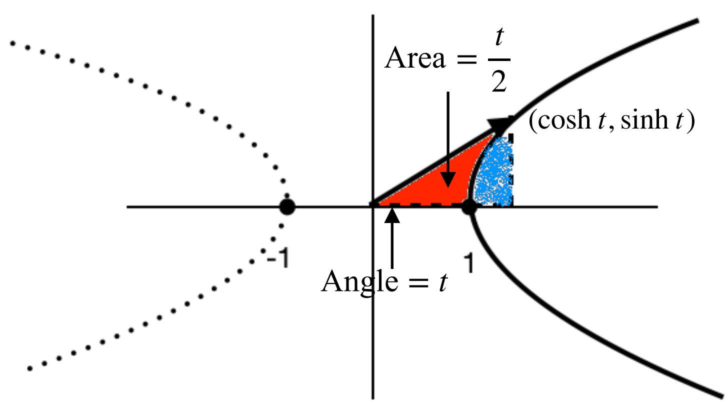


Figure 2

$x^2 - y^2 = 1 \quad x \geq 1$

Note that, in general, $\cosh^2 t - \sinh^2 t = r^2$ and $x = r \cosh t$ where r is a constant representing the distance from the origin to the apex of the hyperbola. In this case, $r = 1$.

$\cosh^2 t - \sinh^2 t = 1 = r$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\begin{aligned} e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned}$$

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ + \; e^{-i\theta} &= \cos\theta - i\sin\theta \\ \hline e^{i\theta} + e^{-i\theta} &= 2\cos\theta \\ \frac{e^{i\theta} + e^{-i\theta}}{2} &= \cos\theta \end{aligned}$$

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ - \; e^{-i\theta} &= \cos\theta - i\sin\theta \\ \hline e^{i\theta} - e^{-i\theta} &= 2i\sin\theta \\ \frac{e^{i\theta} - e^{-i\theta}}{2i} &= \sin\theta \end{aligned}$$

Let $\theta = it$; then

$$\begin{aligned} \cos(it) &= \frac{e^{i(it)} + e^{-i(it)}}{2} \\ &= \frac{e^{-t} + e^{t}}{2} \\ &= \frac{e^{t} + e^{-t}}{2} \end{aligned}$$

$$\begin{aligned} \sin(it) &= \frac{e^{i(it)} - e^{-i(it)}}{2i} \\ &= \frac{e^{-t} - e^{t}}{2i} \\ -\sin(it) &= -\frac{e^{t} - e^{-t}}{2i} \\ -i\sin(it) &= \frac{e^{t} - e^{-t}}{2} \end{aligned}$$

Let $x(t) = \frac{e^{t} + e^{-t}}{2}$; and $y(t) = \frac{e^{t} - e^{-t}}{2}$

We know that, for a hyperbola, $x^2 - y^2 = 1$ so

$$\left(\frac{e^{t} + e^{-t}}{2}\right)^2 - \left(\frac{e^{t} + e^{-t}}{2}\right)^2 = 1$$

$$\begin{aligned} \frac{\left(e^{2t} + 2e^{t}e^{-t} + e^{-2t}\right) - \left(e^{2t} - 2e^{t}e^{-t} + e^{-2t}\right)}{4} &= 1 \\ \frac{e^{2t} + 2e^{t}e^{-t} + \cancel{e^{-2t}} - \cancel{e^{2t}} + 2e^{t}e^{-t} - \cancel{e^{-2t}}}{4} &= 1 \\ \frac{2e^{t}e^{-t} + 2e^{t}e^{-t}}{4} &= 1 \\ \frac{4e^{t}e^{-t}}{4} &= 1 \end{aligned}$$

But $e^t e^{-t} = e^0 = 1$ so

$$\frac{4e^t e^{-t}}{4} = \frac{4 \cdot 1}{4} = 1$$ which, of course, is true

Recall, though, that we defined the x and y coordinates of the hyperbolic function as

x = cosh t and y = sinh t

Therefore,

$$\cosh t = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh t = \frac{e^t - e^{-t}}{2}$$

Finally,

$$\cosh t = \frac{e^t + e^{-t}}{2} = \cos it \quad \text{and} \quad \sinh t = \frac{e^t - e^{-t}}{2} = -i \sin it$$

Thus,

$\cosh(t) = \cos(it)$ and $\sinh(t) = -i\sin(it)$

The area between the x-axis, the left-hand margin of the hyperbola and a line connecting the origin to the point (cosh t, sinh t) on the hyperbola determines the input angle for the hyperbolic trig functions Here, we derive this value.

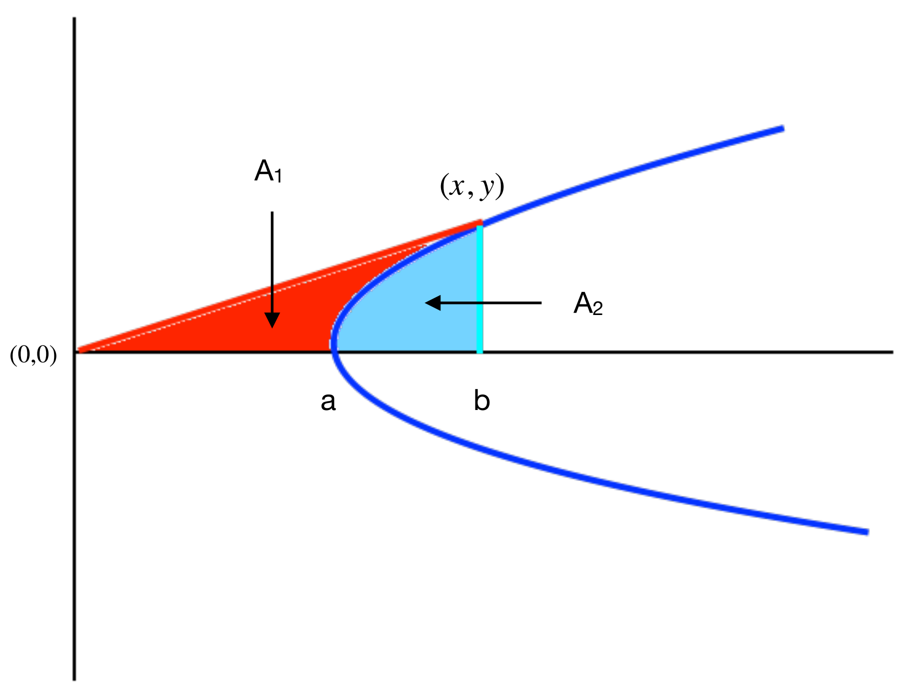


Figure 3

In figure 3, the area that determines the hyperbolic trig function input function is given by A_1. To find this area, we find the area of the triangle from the origin, **(0,0)** to **b** to **(x,y)** and subtract the area of A_2.

The area of triangle is its base times its height. The length of its base is x and its height is y so its area, A_{Tr} is

$$A_{Tr} = A_1 + A_2 = \frac{1}{2}xy \quad \Rightarrow \quad A_1 = A_{Tr} - A_2 = \frac{1}{2}xy - \int_a^b y\,dx$$

$x = \cosh t = \dfrac{e^t + e^{-t}}{2}$ and $y = \sinh t = \dfrac{e^t - e^{-t}}{2}$ so

$$\begin{aligned} A_{Tr} &= \frac{1}{2}\left[\frac{1}{2}\left(e^t + e^{-t}\right)\frac{1}{2}\left(e^t - e^{-t}\right)\right] \\ &= \frac{1}{8}\left(e^t + e^{-t}\right)\left(e^t - e^{-t}\right) \\ &= \frac{1}{8}\left(e^{2t} - e^{-2t}\right) \\ &= \frac{1}{8}e^{2t} - \frac{1}{8}e^{-2t} \end{aligned}$$

Now we need to find A_2. We know that

$$y = \frac{1}{2}\left(e^t - e^{-t}\right) \quad x = \frac{1}{2}\left(e^t + e^{-t}\right) \quad dx = \frac{1}{2}\left(e^t - e^{-t}\right)\,dt$$

so

$$\begin{aligned} \int_a^b y\,dx &= \int_a^b \frac{1}{2}\left(e^t - e^{-t}\right)\frac{1}{2}\left(e^t - e^{-t}\right)\,dt \\ &= \frac{1}{4}\int_a^b e^{2t} - 2e^t e^{-t} + e^{-2t}\,dt \\ &= \frac{1}{4}\int_a^b e^{2t} - 2 + e^{-2t}\,dt \end{aligned}$$

We have to convert our limits of integration from the **x** variables to **t** variables. At $x = a$, our angle is 0 and at $x = b$, our angle is t. Thus, $a = 0$ and $b = t$. Since our limits of integration are in terms of t, we'll change the exponential exponents from t to u. We have

$$
\begin{aligned}
A_2 &= \frac{1}{4}\int_0^t e^{2u} - 2 + e^{-2u}\,du \\
&= \frac{1}{4}\left(\frac{1}{2}e^{2u} - 2u - \frac{1}{2}e^{-2u}\right)\Bigg|_0^t \\
&= \frac{1}{8}e^{2t} - \frac{1}{2}t + \frac{1}{8}e^{-2t} - \frac{1}{8}e^{2\cdot 0} + 2\cdot 0 + \frac{1}{8}e^{-2\cdot 0} \\
&= \frac{1}{8}e^{2t} - \frac{1}{2}t + \frac{1}{8}e^{-2t} - \frac{1}{8}e^{0} + 2\cdot 0 + \frac{1}{8}e^{0} \\
&= \frac{1}{8}e^{2t} - \frac{1}{2}t + \frac{1}{8}e^{-2t}
\end{aligned}
$$

Now we need to put both pieces together:

$$
A_{Tr} = \frac{1}{8}e^{2t} + \frac{1}{8}e^{-2t}
$$

and

$$
A_2 = \frac{1}{8}e^{2t} - \frac{1}{2}t + \frac{1}{8}e^{-2t}
$$

Therefore,

$$
\begin{aligned}
A_1 &= A_{Tr} - A_2 \\
&= \left(\frac{1}{8}e^{2t} + \frac{1}{8}e^{-2t}\right) - \left(\frac{1}{8}e^{2t} - \frac{1}{2}t + \frac{1}{8}e^{-2t}\right) \\
&= \frac{t}{2}
\end{aligned}
$$

Reference

https://www.youtube.com/c/blackpenredpen/search?query=hyperbolic%20functions